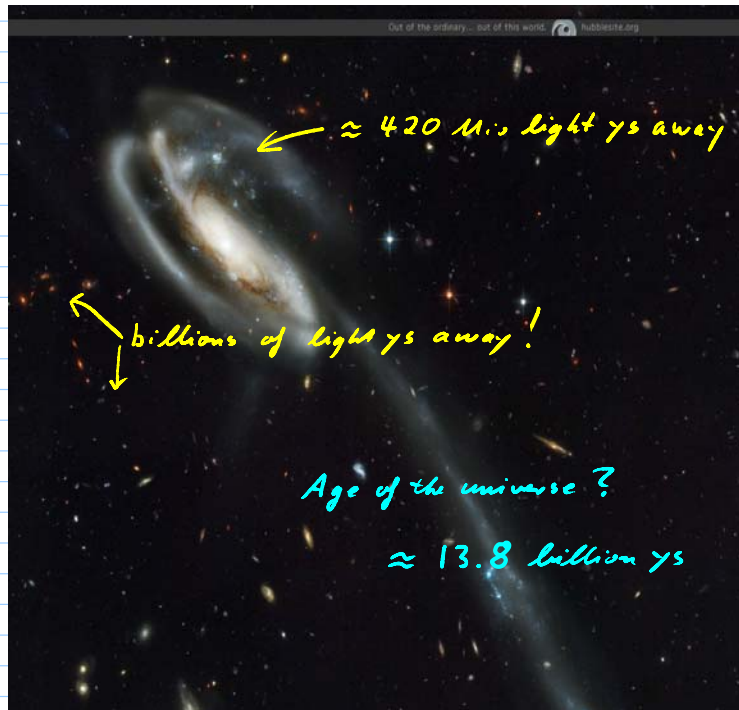
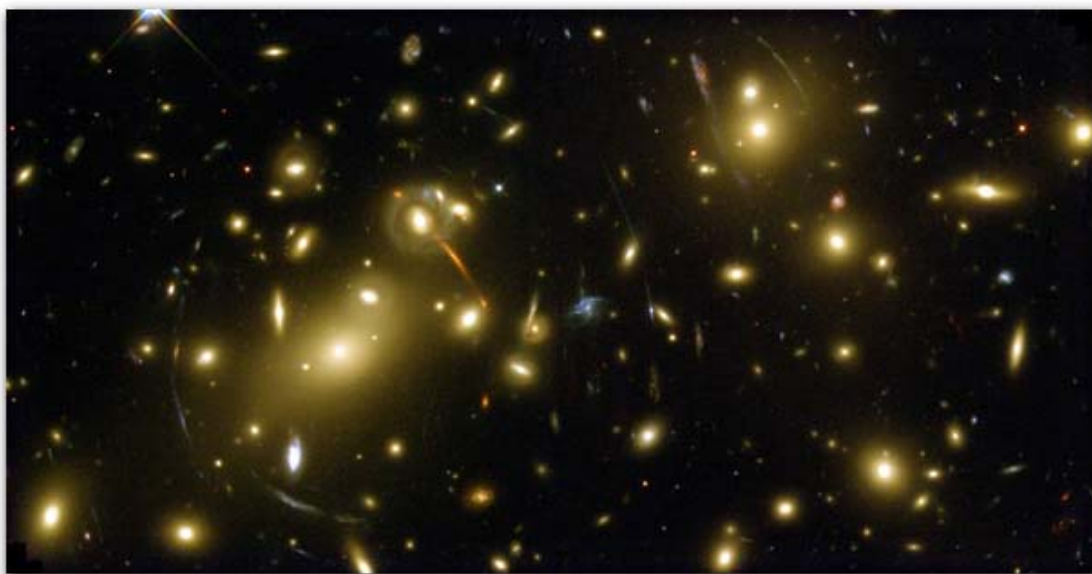


uwaterloo.ca/poi

The 'Tadpole' galaxy



Spacetime's curvature can be seen directly:



HST: ABELL2218

How to describe spacetime?

A. Math

Strategy: $\square$ Start with a mere "set" of points (events), $\mathcal{M}$

Then add structure:

- $\square$ Define open neighborhoods (i.e., a "topology" on $\mathcal{M}$)
- $\square$ Define "separability" of points (i.e. Hausdorff condition)
- $\square$ Define "continuity" (preimage of open sets is open)
- $\square$ Define "differentiability" (via chart change diffeability)

later: $\square$ Define tangent & tensor spaces

$\vdots$

Curvature = nontriviality of parallel transport

Other descriptions of curvature?

(Why consider others? May be useful for quantum gravity b/c what's on previous page is likely over idealized.)

- $\square$ Curvature = sum of angles in triangle $\neq \pi$
- $\square$ Curvature = nontriviality of Pythagoras law
- $\square$ Curvature = tidal forces. Math of it: Sectional curvatures
- $\square$ Curvature $\stackrel{?}{=}$ nontrivial sound of object when vibrating

This field is called Spectral Geometry.

Interesting b/c connects mathematical languages of quantum theory (spectrum etc) and general relativity.

- $\square$ Curvature $\stackrel{?}{=}$ nontrivial entanglement in vacuum fluctuations

B) Structure and properties of General Relativity?

▣ Equations of motion

for scalars, vectors, spinors and curvature

▣ Symmetries

local and global conservation laws, if any!

▣ Tetrad formulation, GR as a gauge theory

▣ Singularities, and their unavoidability

C) Applications to cosmology

▣ Classification of exact solutions

▣ Models of cosmological matter

▣ FRW models, while

using the tetrad formalism

to exercise it. (e.g. for later use in quantum gravity)

▣ Cosmic inflation

▣ Black holes



A. Pseudo-Riemannian Differential Geometry

□ Differentiable Manifolds

(Riemann $\approx$ 1850s, Poincaré $\approx$ 1890s, Whitney $\approx$ 1930s...)

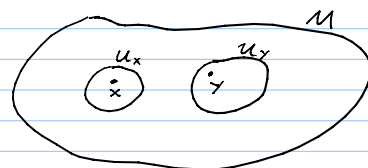
Def: An n -dimensional topological Manifold, M , is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Here:

Def: A topological space, M , is a set, together with a specification of subsets U_i , which will be called "open subsets", which must obey $U_i \cap U_j$ is open, and $\bigcup_i U_i$ is open.

Def: A topological space M is called Hausdorff, if it is separable, i. e., if $x, y \in M$ and $x \neq y$ then x, y are elements of some disjoint open sets.

$\forall x, y: x \neq y \exists U_x, U_y \text{ open: } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \{\}$
↑ "for all" ↑ "there exist"



∅ empty set

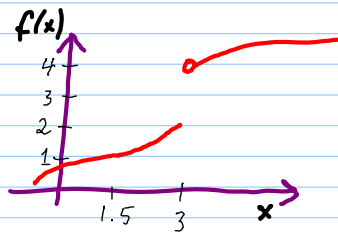
Example: $\mathbb{R}^n$ with its usual definition of open sets.

Now how is the term "homeomorphic" defined?

For this, we need to define "continuity" first:

Recall: If A, B are topol. spaces, then $f: A \rightarrow B$ is called continuous if $(U \subset B \text{ is open} \Rightarrow \underbrace{f^{-1}(U)}_{= \{x \in A : f(x) \in U\}} \subset A \text{ is open})$

Example:



$f: \mathbb{R} \rightarrow \mathbb{R}$

Choose $U := (1, 3)$ open

But $f^{-1}(U) = [1.5, 3]$ not open

Remark: Powerful definition that can be applied very generally.

Why important for us here?

We can now express the idea that a topological Hausdorff space U is continuously parametrizable (as spacetime appears to be)!

Def: Let A, B be topological spaces. Then, a function $f: A \rightarrow B$ is called a homeomorphism, if f^{-1} exists and if both f and f^{-1} are continuous.

Def: We say that A is locally homeomorphic to B if for all $p \in A$ there exists an open neighborhood U_p of p , ($p \in U_p$) which is homeomorphic to an open set in B .

We choose $B := \mathbb{R}^n$:

Recall:

Def: An n -dimensional topological Manifold, M , is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Now how is the term

Differentiable Manifold defined?

Def: A local homeomorphism,

$$h: \mathcal{U} \rightarrow \mathbb{R}^n, \quad \mathcal{U} \subset M$$

$\uparrow$ called "domain"

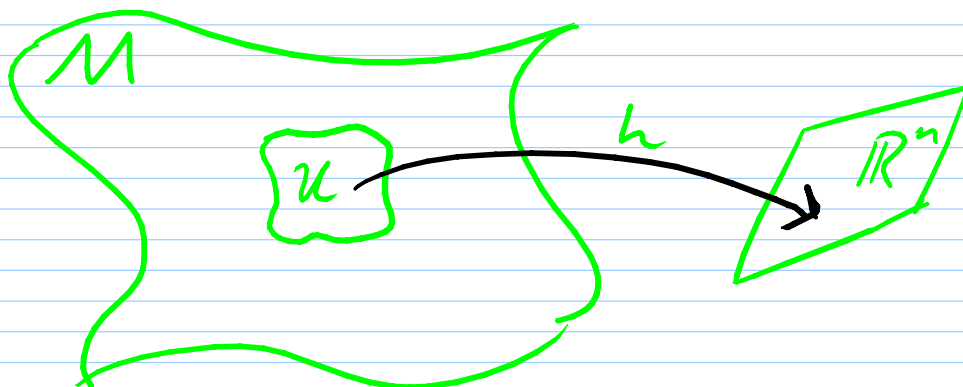
is called a chart of M .

For any point $q \in \mathcal{U}$ its image

$$h(q) \in \mathbb{R}^n$$

is a set of n numbers $(x_1, x_2, \dots, x_n)$ called the coordinates of q .

Def: A chart, h , with domain U ,



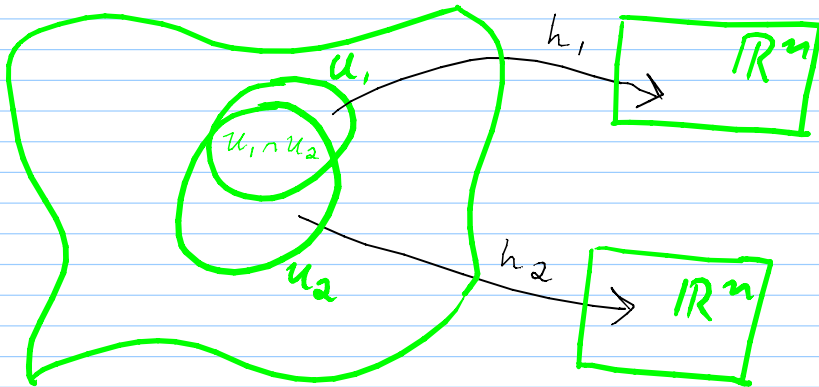
is also called a

local coordinate system for U .

Def: A collection of charts h_α with domains U_α is called an atlas if $\bigcup_\alpha U_\alpha = M$.

→ What, if we want to change coordinates, i.e. if we want to re-label the points of (e.g. a subset of) the manifold?

Consider 2 charts h_1, h_2 , with intersecting domains $U_1 \cap U_2 \neq \emptyset$:



Then, $h_{12} = h_2 \circ h_1^{-1}$ is a continuous change of coordinates map $h_{12}: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Notice: For maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ we know what differentiability means!

Strategy: Let us define the differentiability of an atlas through the differentiability of its chart changes:

Def: An atlas is called C^r differentiable, if all its coordinate changes, $h_{\alpha\beta}$, are C^r diffeomorphisms, i.e., r times continuously differentiable.

Strategy: Enlarge atlas so every point of M is in multiple charts.

Then, diffability of M is definable through atlas diffability

Def: Given a C^r differentiable atlas, A , we can generate a maximal C^r differentiable atlas, $D(A)$, by adding all charts whose chart changes with charts in A are differentiable.

Def: $D(A)$ is also called a "Differentiable Structure" of class C^r for M .

Def: A differentiable manifold of class C^r is a topol. manifold with a maximal atlas of class C^r , i.e., with a differentiable structure of class C^r .

Theorem: (Whitney)

Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

I.e. any diffable structure can be smoothed. Any lack of higher diffability is due to unlucky choice of chart.

Def: Since any C^1 manifold is also a C^∞ manifold, we also call diffable manifolds simply smooth manifolds.

